MCSHANE-WHITNEY EXTENSIONS IN CONSTRUCTIVE ANALYSIS

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ABSTRACT. Within Bishop-style constructive mathematics we study the classical McShane-Whitney theorem on the extendability of real-valued Lipschitz functions defined on a subset of a metric space. Using a formulation similar to the formulation of McShane-Whitney theorem, we show that the Lipschitz real-valued functions on a totally bounded space are uniformly dense in the set of uniformly continuous functions. Through the introduced notion of a McShane-Whitney pair we describe the constructive content of the original McShane-Whitney extension and we examine how the properties of a Lipschitz function defined on the subspace of the pair extend to its McShane-Whitney extensions on the space of the pair. Similar McShane-Whitney pairs and extensions are established for Hölder functions and ν -continuous functions, where ν is a modulus of continuity. A Lipschitz version of a fundamental corollary of the Hahn-Banach theorem, and the approximate McShane-Whitney theorem are shown.

Introduction

A central theme in mathematical analysis, classical or constructive, is the extension of a continuous function f, defined on a subspace A of a space X and satisfying some property P, to a continuous function f^* defined on X and satisfying the property P. A "general extension theorem" is a statement providing conditions on X and A such that such an extension of f to f^* is possible. As it is noted in [11], p. 149, "general extension theorems are rare and usually have interesting topological consequences".

If X is a normal (and Hausdorff) topological space, A is a closed subspace of X, the continuous function on A takes values in \mathbb{R} , and P(f) is "f is bounded", the corresponding general extension theorem is Tietze's extension theorem, which is actually a characterisation of normality (see [11], pp. 149-151) and its proof rests on Urysohn's lemma. If X is a metric space, hence normal, we get the restriction of the classical Tietze extension theorem to metric spaces. If X is a complete and separable metric space, Urysohn's lemma and Tietze's extension theorem are provable in RCA₀ (see [23], pp. 90-91). Within Bishop-style constructivism, which is the framework of constructive mathematics we work within, a Tietze extension theorem for metric spaces is proved in [4], p. 120. According to it, if X is a

¹Note that the general pattern of an extension theorem is not to bestow extra properties upon f.



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^{*} This paper is a major extension of [22].

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metric space, A is a locally compact² subset of X, and $f: A \to [a, b]$ is continuous i.e., it is uniformly continuous on every bounded subset of A (see [4], p. 110), there is an extension function $f^*: X \to [a, b]$ of f on X, which is also uniformly continuous on every bounded subset of X. As it is customary in the constructive treatment of a classical extension theorem, the constructive theorem is less general, and sometimes the notions involved have to be understood differently. The benefit of the constructive approach though, is that the, usually hidden, computational content of the classical theorem is revealed.

If X is a normed space, A is a linear subspace of X and the property P(f) for the continuous function $f:A\to\mathbb{R}$ is "f is linear and ||f||=a", where $a\in[0,+\infty)$, the corresponding extension theorem is (one version of) the Hahn-Banach theorem. Its proof is non-constructive, as it rests on the axiom of choice. Constructively, one needs to add some extra properties on X, in order to avoid the axiom of choice. E.g., in [13] the normed space X is also uniformly convex, complete and its norm is Gâteaux differentiable. Since constructively the existence of the norm of the continuous functional is equivalent to the locatedness of its kernel (see [7], pp. 53-54), in the formulation of the constructive Hahn-Banach threorem we need to suppose that the norm ||f|| of f exists. Then the norm $||f^*||$ of the extension function f^* exists too, and $||f^*|| = ||f||$. The computational content of the classical theorem is revealed in the constructively provable, approximate version of the Hahn-Banach theorem (see [6], pp. 39-40). According to it, if X is a separable normed space and ||f|| exists, then for every $\epsilon > 0$ there is a continuous and linear extension f^* of f with a norm $||f^*||$ satisfying

$$||f^*|| \le ||f|| + \epsilon.$$

Hence, in the approximate version of the Hahn-Banach theorem the numerical part of P(f) is satisfied only approximately.

If X is a metric space, A is any subspace of X and the property P(f) for the (necessarily continuous) function $f: A \to \mathbb{R}$ is "f is Lipschitz", the corresponding extension theorem is the McShane-Whitney theorem, which appeared first³ in [19] and [26]. This theorem admits a classical proof that uses the axiom of choice, and it is very similar to the proof of the analytic version of the Hahn-Banach theorem (see [9], pp. 224-5).

Following [25], pp.16-17, if $\sigma > 0$ and f is σ -Lipschitz i.e., $\forall_{a,a' \in A} (|f(a) - f(a')| \le \sigma d(a,a'))$, and if $x \in X \setminus A$, a σ -Lipschitz extension g of f on $A \cup \{x\}$ must satisfy

$$\forall_{a \in A} (|g(x) - g(a)| = |g(x) - f(a)| \le \sigma d(x, a)),$$

which is equivalent to

$$\forall_{a \in A} (f(a) - \sigma d(x, a) \le g(x) \le f(a) + \sigma d(x, a)).$$

Since $\forall_{a,a'\in A}(|f(a)-f(a')| \leq \sigma d(a,a') \leq \sigma(d(a,x)+d(x,a')))$, we get

$$\forall_{a,a' \in A} (f(a) - \sigma d(a, x) \le f(a') + \sigma d(x, a')).$$

Hence, if we fix some $a \in A$ (the case $A = \emptyset$ is trivial), then for all $a' \in A$ we get

$$f(a) - \sigma d(a, x) \le f(a') + \sigma d(x, a'),$$

²In Bishop-style constructive analysis a locally compact metric space X is a metric space such that every bounded subset of X is included in a compact subset of X.

 $^{^{3}}$ To determine metric spaces X and Y, such that a similar extension theorem for Y-valued Lipschitz functions defined on a subset A of X holds, is a non-trivial problem under active current study (see [2], [5] and [24]).

therefore, using the classical completeness property of real numbers, we get

$$f(a) - \sigma d(x, a) \le \inf\{f(a') + \sigma d(x, a') \mid a' \in A\}.$$

Since a is an arbitrary element of A, we conclude that

$$s_x \equiv \sup\{f(a) - \sigma d(x, a) \mid a \in A\} \le \inf\{f(a) + \sigma d(x, a) \mid a \in A\} \equiv i_x.$$

If we define g(x) to be any value in the interval $[s_x, i_x]$, we get the required extension of f on $A \cup \{x\}$. If we apply Zorn's lemma on the non-empty poset

$$\{(B,h)\mid B\supseteq A\ \&\ h:B\to\mathbb{R}\ \text{is }\sigma\text{-Lipschitz}\ \&\ h_{|A}=f\},$$

where

$$(B_1, h_1) \le (B_2, h_2) :\Leftrightarrow B_1 \subseteq B_2 \& h_{2|B_1} = h_1,$$

its maximal element is the required σ -Lipschitz extension of f.

What McShane and Whitney independently observed though, was that if one defines $g^*(x) = s_x$ and $g^*(x) = i_x$, for all $x \in X$, the resulting functions g^* and $g^*(x) = i_x$, for all $g^*(x) = i_x$, the resulting functions $g^*(x) = i_x$, are $g^*(x) = i_x$, for all $g^*(x) = i_x$, the resulting functions $g^*(x) = i_x$, are $g^*(x) = i_x$, and $g^*(x) = i_x$, and $g^*(x) = i_x$, the probability expected by the above possible choice of $g^*(x) = i_x$, and $g^*(x) = i_x$, the probability expected by the above possible choice of $g^*(x) = i_x$, and $g^*(x) = i_x$, the resulting functions $g^*(x) = i_x$, and $g^*(x) = i_x$, and $g^*(x) = i_x$, and $g^*(x) = i_x$, the resulting functions $g^*(x) = i_x$, and $g^*($

If X is a metric space and A is a subset of X, we say that (X, A) is a McShane-Whitney pair, if the functions g^* and g^* are well-defined (Definition 2.1). Classically, all pairs (X, A), where $A \neq \emptyset$, are McShane-Whitney pairs. The definitions of g^* and g^* though, depend on the existence of the supremum and infimum of non-empty, bounded above and bounded below subsets of reals, respectively. Since the classical completeness property of reals implies constructively the principle of the excluded middle (PEM) (see [7], p. 32), it is not a surprise that with a similar argument we can show that the statement "every pair (X, A) is a McShane-Whitney pair" implies constructively the principle PEM (Remark 2.2).

In this paper we study the McShane-Whitney extension within Bishop's informal system of constructive mathematics BISH (see [3], [4], [6] and [7]). A formal system for BISH is Myhill's system CST with dependent choice (see [20]), or Aczel's system of constructive set theory CZF with dependent choice (see [1]).

The main results included here reveal, in our view, the deep interactions between the McShane-Whitney theorem and the Hahn-Banach theorem. We can summarise our results as follows:

- If X is a totally bounded metric space, we prove the uniform density of the Lipschitz functions on X in the set of uniformly continuous functions on X (Corollary 1.1). For that we prove Theorem 1.5 that has a formulation similar to the formulation of the McShane-Whitney theorem (Theorem 2.3).
- We introduce the notion of a McShane-Whitney pair that reveals the constructive content of the classical McShane-Whitney extension in Theorem 2.3.
- We describe how properties of f, like boundedness, existence of the Lipschitz constant L(f) of f, linearity, and convexity, extend to f^* and f^* (subsections 2.1 and 3.1).
- We provide conditions on X and/or Y so that (X, A) is a McShane-Whitney pair, from the constructive point of view (Proposition 2.1).
- We define a local version of a McShane-Whitney pair (subsection 2.2).
- We prove a Lipschitz version of a fundamental corollary of the Hahn-Banach theorem (Theorem 3.4).

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- We show how the McShane-Whitney pairs and the McShane-Whitney construction can be applied to Hölder functions and ν -continuous functions, where ν is a modulus of continuity (subsection 2.3).
- We prove the approximate McShane-Whitney theorem for Lipschitz functions defined on a finite-dimensional subspace of a normed space (Theorem 3.5). This is, in our view, the most technically interesting result of this paper.

First we prove some fundamental facts in the constructive theory of Lipschitz functions, which is quite different from its classical counterpart.

1. Lipschitz functions, constructively

Constructive Lipschitz analysis is quite underdeveloped. Some related results are found scattered e.g., in [10], [15], [17], [8]. For the sake of completeness, we include the definitions of some fundamental concepts.

Definition 1.1. Let $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$. If A is bounded above, we define

$$\begin{split} b \geq A :&\Leftrightarrow \forall_{a \in A} (b \geq a), \\ [A) :&= \{c \in \mathbb{R} \mid c \geq A\}, \\ l :&= \sup A :&\Leftrightarrow l \geq A \ \& \ \forall_{\epsilon > 0} \exists_{a \in A} (a > l - \epsilon), \\ \lambda :&= \mathrm{lub} A :&\Leftrightarrow \lambda \geq A \ \& \ \forall_{b \in [A)} (b \geq \lambda). \end{split}$$

If A is bounded below, $b \leq A$, (A], $m := \inf A$, and $\mu := \operatorname{glb} A$ are defined in a dual way.

According to the classical completeness of \mathbb{R} , if $A \subseteq \mathbb{R}$ is inhabited and bounded above, then $\sup A$ exists. Constructively, this statement implies the principle of the excluded middle PEM (i.e., the scheme $P \vee \neg P$) (see [7], p. 32). A similar argument⁴ shows that the corresponding existence of $\operatorname{lub} A$ implies the weak principle of the excluded middle WPEM (i.e., the scheme $\neg P \vee \neg \neg P$). It is immediate to show that if $\sup A$ exists, then $\operatorname{lub} A$ exists and they are equal, for every $A \subseteq \mathbb{R}$. The converse is not generally true (see [18], p. 27).

Note that Bishop used the terms lubA and supA (and similarly the terms glbA and infA) without distinction. In [18] Mandelkern gave reasons to keep both notions at work. In [18], pp. 24-25, he also gave a necessary and sufficient condition for the existence of lubA and glbA, similar to the constructive version of completeness given by Bishop and Bridges in [4], p. 37, and he proved the following remark: If $A \subseteq \mathbb{R}$ is bounded and glbA exists, then sup(A] exists and sup(A) = glbA, while if lubA exists, then inf[A) exists and inf[A) = lubA.

Definition 1.2. If X, Y are sets, we denote by $\mathbb{F}(X, Y)$ the set of functions of type $X \to Y$ and by $\mathbb{F}(X)$ the set of functions of type $X \to \mathbb{R}$. If $a \in \mathbb{R}$, then \overline{a}_X denotes the constant map in $\mathbb{F}(X)$ with value a, and $\mathrm{Const}(X)$ is the set of constant maps on X. If $(X, d), (Y, \rho)$ are metric spaces, then $C_u(X, Y)$ denotes the set of uniformly continuous functions from X to Y, where $f \in C_u(X, Y)$, if there is a function $\omega_f : \mathbb{R}^+ \to \mathbb{R}^+$, a modulus of uniform continuity of f, such that for every $\epsilon > 0$

$$\forall_{x,y \in X} (d(x,y) \le \omega_f(\epsilon) \Rightarrow \rho(f(x), f(y)) \le \epsilon).$$

⁴This argument goes as follows. If $A := \{0\} \cup \{x \in \mathbb{R} \mid x = 1 \& P\}$ and $\lambda := \text{lub}A$ exists, then by the constructive version of the classical trichotomy of reals (Corollary (2.17) in [4], p. 26) we have that $\lambda > 0$ or $\lambda < 1$. If $\lambda > 0$, we suppose $\neg P$, hence $A = \{0\}$, and λ cannot be the least upper bound of $\{0\}$. Hence we showed $\neg \neg P$. If $\lambda < 1$, we suppose P, hence $A = \{0, 1\}$ and we get the contradiction $1 \le \lambda < 1$. Hence we showed $\neg P$.

We denote by $C_u(X)$ the set $C_u(X, \mathbb{R})$, where \mathbb{R} is equipped with its standard metric. The metric d_{x_0} at the point $x_0 \in X$ is defined by $d_{x_0}(x) := d(x_0, x)$, for every $x \in X$, and $U_0(X) := \{d_{x_0} \mid x_0 \in X\}$. The set X_0 of the d-distinct pairs of X is defined by

$$X_0 := \{(x, y) \in X \times X \mid d(x, y) > 0\}.$$

The Lipschitz functions are defined constructively as in the classical setting. Throughout this section, X is a set equipped with a metric d, and Y is a set equipped with a metric ρ .

Definition 1.3. The set of Lipschitz functions from X to Y is defined by

$$\operatorname{Lip}(X,Y) := \bigcup_{\sigma \ge 0} \operatorname{Lip}(X,Y,\sigma),$$

$$\operatorname{Lip}(X, Y, \sigma) := \{ f \in \mathbb{F}(X, Y) \mid \forall_{x, y \in X} (\rho(f(x), f(y)) \le \sigma d(x, y)) \}.$$

If $Y = \mathbb{R}$, we write Lip(X) and $\text{Lip}(X, \sigma)$, respectively.

Clearly, $\operatorname{Lip}(X,Y) \subseteq C_u(X,Y)$. An element of $\operatorname{Lip}(X,Y)$ sends a bounded subset of X to a bounded subset of Y, which is not generally the case for elements of $C_u(X,Y)$; the identity function $\operatorname{id}: \mathbb{N} \to \mathbb{R}$, where \mathbb{N} is equipped with the discrete metric, is in $\operatorname{C}_u(\mathbb{N}) \setminus \operatorname{Lip}(\mathbb{N})$ and $\operatorname{id}(\mathbb{N}) = \mathbb{N}$ is unbounded in \mathbb{R} . The following proposition is easy to show.

Proposition 1.4. The set Lip(X) includes the sets $U_0(X)$, Const(X), and it is closed under addition and multiplication by reals. If every element of $C_u(X)$ is a bounded function, then Lip(X) is closed under multiplication.

Next we show⁵ constructively the uniform density of $\operatorname{Lip}(X)$ in $C_u(X)$, in case X is totally bounded i.e., if there is a finite ϵ -approximation to X, for every $\epsilon > 0$ (see [4], p. 94). The formulation and the proof of the following theorem are analogous to the formulation and the proof of Theorem 2.3.

Theorem 1.5. Let (X,d) be totally bounded. If $f \in C_u(X)$ and $\epsilon > 0$, there are $\sigma > 0$, $g^* \in \text{Lip}(X,\sigma)$ and $^*g \in \text{Lip}(X,\sigma)$ such that the following hold:

- (i) $f \epsilon \le g^* \le f \le *g \le f + \epsilon$.
- (ii) For every $e \in \text{Lip}(X, \sigma)$, $e \leq f \Rightarrow e \leq g^*$.
- (iii) For every $e \in \text{Lip}(X, \sigma)$, $f \leq e \Rightarrow *g \leq e$.
- (iv) The pair $(g^*, *g)$ is the unique pair of functions in $Lip(X, \sigma)$ satisfying conditions (i)-(iii).

Proof. (i) Let ω_f be a modulus of uniform continuity of f, and $M_f > 0$ a bound of f. We define the functions $h_x : X \to \mathbb{R}$ and $g^* : X \to \mathbb{R}$ by

$$h_x := f + \sigma d_x,$$

$$\sigma := \frac{2M_f}{\omega_f(\epsilon)} > 0,$$

$$g^*(x) := \inf\{h_x(y) \mid y \in X\} = \inf\{f(y) + \sigma d(x, y) \mid y \in X\},\$$

for every $x \in X$. Note that $g^*(x)$ is well-defined, since $h_x \in C_u(X)$ and the infimum of h_x exists, since X is totally bounded (see [4], p.94 and p.38). First we show that $g \in \text{Lip}(X, \sigma)$. If $x_1, x_2, y \in X$, the inequality $d(x_1, y) \leq d(x_2, y) + d(x_1, x_2)$ implies that $f(y) + \sigma d(x_1, y) \leq (f(y) + \sigma d(x_2, y)) + \sigma d(x_1, x_2)$, therefore

⁵This proof appeared first in [21].

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 $g^*(x_1) \leq g^*(x_2) + \sigma d(x_1, x_2)$, or $g^*(x_1) - g^*(x_2) \leq \sigma d(x_1, x_2)$. Starting with the inequality $d(x_2, y) \leq d(x_1, y) + d(x_1, x_2)$ and working similarly we get that $g^*(x_2) - g^*(x_1) \leq \sigma d(x_1, x_2)$, therefore $|g^*(x_1) - g^*(x_2)| \leq \sigma d(x_1, x_2)$. Next we show that

$$\forall_{x \in X} (f(x) - \epsilon \le g^*(x) \le f(x)).$$

Since $f(x) = f(x) + \sigma d(x, x) = h_x(x) \ge \inf\{h_x(y) \mid y \in X\} = g^*(x)$, for every $x \in X$, we have that $g^* \le f$. Next we show that $\forall_{x \in X} (f(x) - \epsilon \le g^*(x))$. For that we fix $x \in X$ and we show that $\neg (f(x) - \epsilon > g^*(x))$. Note that if $A \subseteq \mathbb{R}, b \in \mathbb{R}$, then $b > \inf A \Rightarrow \exists_{a \in A} (a < b)$. Therefore,

$$f(x) - \epsilon > g^*(x) \Leftrightarrow f(x) - \epsilon > \inf\{f(y) + \sigma d(x, y) \mid y \in X\}$$

$$\Rightarrow \exists_{y \in X} (f(x) - \epsilon > f(y) + \sigma d(x, y))$$

$$\Leftrightarrow \exists_{y \in X} (f(x) - f(y) > \epsilon + \sigma d(x, y)).$$

For this y we show that $d(x,y) \leq \omega_f(\epsilon)$. If $d(x,y) > \omega_f(\epsilon)$, we have that

$$2M_f \ge f(x) + M_f \ge f(x) - f(y) > \epsilon + 2M_f \frac{d(x,y)}{\omega_f(\epsilon)} > \epsilon + 2M_f > 2M_f,$$

which is a contradiction. Hence, by the uniform continuity of f we get that $|f(x) - f(y)| \le \epsilon$, therefore the contradiction $\epsilon > \epsilon$ is reached, since

$$\epsilon \ge |f(x) - f(y)| \ge f(x) - f(y) > \epsilon + \sigma d(x, y) \ge \epsilon.$$

Next we define the functions $h_x^*: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ by

$$h_x^* := f - \sigma d_x$$

$$^*g(x) := \sup\{h_x^*(y) \mid y \in X\} = \sup\{f(y) - \sigma d(x, y) \mid y \in X\},\$$

for every $x \in X$, and $\sigma = \frac{2M_f}{\omega_f(\epsilon)}$. Similarly⁷ to the proof for g^* , we get that $g^* \in \text{Lip}(X, \sigma)$ and

$$\forall_{x \in X} (f(x) \le {}^*g(x) \le f(x) + \epsilon).$$

- (ii) Let $e \in \text{Lip}(X, \sigma)$ such that $e \leq f$. If we fix some $x \in X$, then for every $y \in X$ we have that $e(x) e(y) \leq |e(x) e(y)| \leq \sigma d(x, y)$, hence $e(x) \leq e(y) + \sigma d(x, y) \leq f(y) + \sigma d(x, y)$, therefore $e(x) \leq g^*(x)$.
- (iii) Let $e \in \text{Lip}(X, \sigma)$ such that $f \leq e$. If we fix some $x \in X$, then for every $y \in X$ we have that $e(y) e(x) \leq |e(y) e(x)| \leq \sigma d(x, y)$, hence $f(y) \sigma d(x, y) \leq e(y) \sigma d(x, y) \leq e(x)$, therefore $*g(x) \leq e(x)$.
- (iv) Let $h^*, h^* \in \text{Lip}(X, \sigma)$ such that the following hold:
- (i) $f \epsilon \le h^* \le f \le h^* \le f + \epsilon$.
- (ii) For every $e \in \text{Lip}(X, \sigma)$, $e < f \Rightarrow e < h^*$.
- (iii) For every $e \in \text{Lip}(X, \sigma)$, $f \leq e \Rightarrow *h \leq e$.

Since $h^* \leq f$, we get by (ii) that $h^* \leq g^*$. Since $g^* \leq f$, we get by (ii) that $g^* \leq h^*$. Using similarly (iii) and (iii), we get $h^* = g$.

From Theorem 1.5(i) we get immediately the following corollary.

⁶By the definition of inf A in [4], p.37, we have that $\forall_{\epsilon>0}\exists_{a\in A}(a<\inf A+\epsilon)$, therefore if $b>\inf A$ and $\epsilon=b-\inf A>0$ we get that $\exists_{a\in A}(a<\inf A+(b-\inf A)=b)$.

⁷To show that $\neg(g^*(x) > f(x) + \epsilon)$ we just use the fact that if $A \subseteq \mathbb{R}$, $b \in \mathbb{R}$, then $\sup A > b \Rightarrow \exists_{a \in A} (a > b)$. The function *g is mentioned in [27], where non-constructive properties of the classical $(\mathbb{R}, <)$ are used.

Corollary 1.6. If X is totally bounded, then Lip(X) is uniformly dense in $C_u(X)$.

The constructive behavior of Lipschitz functions is different from the classical one. For example, Lebesgue's theorem on the almost everywhere differentiability of a Lipschitz function $f:(a,b)\to\mathbb{R}$, and hence Rademacher's generalization (see [12], p. 18), do not hold constructively; Demuth has shown in [10] that in the recursive variety RUSS of BISH (see [6]) there is a Lipschitz function $f:[0,1]\to\mathbb{R}$, which is nowhere differentiable. As we explain in the rest of this paper, there are important differences between the classical and the constructive theory of Lipschitz functions at an even more elementary level.

Definition 1.7. If $f \in \mathbb{F}(X,Y)$, we define the following sets:

$$\Lambda(f) := \{ \sigma \ge 0 \mid \forall_{x,y \in X} (\rho(f(x), f(y)) \le \sigma d(x, y)) \},
\Xi(f) := \{ \sigma \ge 0 \mid \forall_{x,y \in X} (\rho(f(x), f(y)) \ge \sigma d(x, y)) \},
M_0(f) := \{ \sigma_{x,y}(f) \mid (x, y) \in X_0 \},
\sigma_{x,y}(f) := \frac{\rho(f(x), f(y))}{d(x, y)}.$$

Proposition 1.8. If $f \in \mathbb{F}(X,Y)$, then $\Lambda(f) = [M_0(f))$ and $\Xi(f) = (M_0(f)]$.

Proof. First we show that $\Lambda(f) = [M_0(f))$. If $b \geq M_0(f)$, we show that $b \in \Lambda(f)$ i.e., $\forall x,y \in X (\rho(f(x),f(y)) \leq bd(x,y))$. Since $b \geq 0$, let $x,y \in X$ such that $\rho(f(x),f(y)) > bd(x,y)$. Suppose next that d(x,y) > 0, hence $\sigma_{x,y}(f) \in M_0(f)$, and since $b \geq M_0(f)$, we get that $\sigma_{x,y}(f) \leq b \Leftrightarrow \rho(f(x),f(y)) \leq bd(x,y)$ which together with our initial hypothesis $bd(x,y) < \rho(f(x),f(y))$ leads to a contradiction. Hence, $d(x,y) \leq 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$, and our initial hypothesis is reduced to 0 > 0. The converse implication, $\sigma \in \Lambda(f) \Rightarrow \sigma \geq M_0(f)$ follows immediately. The proof of the equality $\Xi(f) = (M_0(f)]$ is similar.

By the classical completeness of \mathbb{R} , if $f \in \operatorname{Lip}(X,Y)$, then $\inf \Lambda(f)$ exists. Constructively, we don't have in general the existence of $\inf \Lambda(f)$. Actually, the statement " $\inf \Lambda(f)$ exists, for every $f \in \operatorname{Lip}(A)$ and every $A \subseteq \mathbb{R}$ " implies WPEM. To see this, we work as follows. If P is a syntactically well-formed proposition, let $A := \{0\} \cup \{x \in \mathbb{R} \mid x = 1 \ \& P\}$. If $f : A \to \mathbb{R}$ is defined by f(a) := a, for every $a \in A$, then $f \in \operatorname{Lip}(A,1)$ and $1 \in \Lambda(f)$. If $m := \inf \Lambda(f)$, then by the constructive version of the classical trichotomy of reals (Corollary (2.17) in [4], p. 26) we have that m > 0 or m < 1. If m > 0, we suppose $\neg P$, hence $A = \{0\}$. In this case $0 \in \Lambda(f)$, and we have that $m \le 0 < m$, which is a contradiction, hence we showed $\neg \neg P$. If m < 1, we suppose P, hence $A = \{0,1\}$. By the definition of $\inf \Lambda(f)$ we can find $\epsilon > 0$ and $\sigma \in \Lambda(f)$ such that $\sigma < m + \epsilon < 1$. Hence, $1 = |0 - 1| \le \sigma |0 - 1| = \sigma < 1$, which is a contradiction. Hence we showed $\neg P$.

Classically, if inf $\Lambda(f)$ exists, then $\sup M_0(f)$ exists and it is equal to $\inf \Lambda(f)$. Constructively, from the existence of $\inf \Lambda(f)$ we only get the existence of $\lim M_0(f)$. If $\sup M_0(f)$ exists though, we can infer constructively the existence of $\inf \Lambda(f)$.

Proposition 1.9. Let $f \in \text{Lip}(X, Y)$.

- (i) If $\sup M_0(f)$ exists, then $\inf \Lambda(f)$ exists and $\inf \Lambda(f) = \min \Lambda(f) = \sup M_0(f)$.
- (ii) If $\inf \Lambda(f)$ exists, then $\operatorname{lub} M_0(f)$ exists and $\operatorname{lub} M_0(f) = \inf \Lambda(f)$.
- (iii) If $lub M_0(f)$ exists, then $inf \Lambda(f)$ exists and $inf \Lambda(f) = lub M_0(f)$.

Proof. (i) Since for $l = \sup M_0(f)$ we have that $l \ge M_0(f)$, by Proposition 1.8 we get that $l \in \Lambda(f)$. Next we show that $l \le \Lambda(f)$. If $\sigma \in \Lambda(f)$, then $\sigma \ge M_0(f)$, therefore $\sigma \ge l$, since the supremum of a set is its least upper bound. Since $l \in \Lambda(f)$, if $\epsilon > 0$, then $l < l + \epsilon$, hence

 $l = \inf \Lambda(f)$.

(ii) If $v = \inf \Lambda(f)$, then $v \geq M_0(f)$, as we have explained in (i). Suppose next that $b \geq M_0(f)$. We show that $b \geq v$, by supposing b < v and reaching a contradiction. Since $b \geq M_0(f)$, by Proposition 1.8 we have that $b \in \Lambda(f)$, and consequently $v \leq b < v$.

(iii) By Proposition 1.8 we have that $\Lambda(f) = [M_0(f))$, hence by Mandelkern's remark after Definition 1.1 we get that $\text{lub}M_0(f) = \inf[M_0(f)) = \inf \Lambda(f)$.

Definition 1.10. Let $f \in \text{Lip}(X, Y)$. If $\sup M_0(f)$ exists, we call $L(f) := \sup M_0(f)$ the Lipschitz constant of f. If $\lim M_0(f)$ exists, we call $L^*(f) := \lim M_0(f)$ the weak Lipschitz constant of f.

Constructively, L(f) does not generally exist. Actually, the statement "L(f) exists, for every $f \in \text{Lip}(l_2)$ " implies the limited principle of omniscience (LPO), where LPO is a weak form of PEM, and $l_2 := \{(x_n) \in \mathbb{F}(\mathbb{N}, \mathbb{R}) \mid \sum_{n \in \mathbb{N}} |x_n|^2 \in \mathbb{R}\}$ is equipped with the 2-norm. This is explained as follows. A linear functional on a normed space X is bounded, if there is $\sigma > 0$ such that $|f(x)| \le \sigma ||x||$, for every $x \in X$ (see [7], p. 51). By the linearity of f we then get $f \in \text{Lip}(X, \sigma)$. Since in this case it is easy to show that $M_0(f) = \{|f(x)| \mid ||x|| = 1\}$, the existence of $\sup M_0(f)$ is equivalent to the existence of ||f||. Hence, the statement "L(f) exists, for every $f \in \text{Lip}(l_2)$ " implies the statement "every bounded linear functional on l_2 is normable". The last statement implies LPO (see [14]).

If L(f) exists, and since $L(f) \ge M_0(f)$, by Proposition 1.8 we get $\forall_{x,y \in X} (\rho(f(x), f(y)) \le L(f)d(x,y))$, i.e., $f \in \text{Lip}(X,Y,L(f))$. If $L^*(f)$ exists, then proceeding similarly we get $\forall_{x,y \in X} (\rho(f(x), f(y)) \le L^*(f)d(x,y))$ i.e., $f \in \text{Lip}(X,Y,L^*(f))$.

Proposition 1.11. Let $A \subseteq X$, $f \in \text{Lip}(A,Y)$ and $g \in \mathbb{F}(X,Y)$ such that

$$g_{|A} = f \& \forall_{\sigma > 0} (f \in \text{Lip}(A, Y, \sigma) \Rightarrow g \in \text{Lip}(X, Y, \sigma)).$$

- (i) If L(f) exists, then L(g) exists and L(g) = L(f).
- (ii) If $L^*(f)$ exists, then $L^*(g)$ exists and $L^*(g) = L^*(f)$.

Proof. (i) If $L(f) = \sup M_0(f)$ we show that $L(f) = \sup M_0(g)$, where $M_0(f) = \{\sigma_{a,b}(f) \mid (a,b) \in A_0\}$, $M_0(g) = \{\sigma_{x,y}(g) \mid (x,y) \in X_0\}$, and $M_0(f) \subseteq M_0(g)$. First we show that $L(f) \geq M_0(g)$. Since $L(f) \geq 0$ and $f \in \text{Lip}(A,Y,L(f))$, we get by hypothesis that g is also in Lip(X,Y,L(f)), therefore $L(f) \geq M_0(g)$. If $\epsilon > 0$, there exists $\sigma_{a,b}(f) \in M_0(f)$ such that $\sigma_{a,b}(f) > L(f) - \epsilon$, which implies that $\sigma_{a,b}(f) = \sigma_{a,b}(g) > L(f) - \epsilon$, and $\sigma_{a,b}(g) \in M_0(g)$, since $A_0 \subseteq X_0$.

(ii) If $L^*(f) = \text{lub} M_0(f)$ we show that $L^*(f) = \text{lub} M_0(g)$. First we show that $L^*(f) \geq M_0(g)$. As in (i), since $L^*(f) \geq 0$ and $f \in \text{Lip}(A, Y, L^*(f))$, we get by hypothesis that g is also in $\text{Lip}(X, Y, L^*(f))$, therefore $L^*(f) \geq M_0(g)$. If $m \geq M_0(g)$, then $m \geq M_0(f)$, since $M_0(g) \subseteq M_0(g)$, therefore $m \geq L^*(f)$.

Note that if $f \in \text{Lip}(A, Y)$, $g \in \text{Lip}(X, Y)$ such that L(f), L(g) exist and L(f) = L(g), then it is immediate to see that $\forall_{\sigma \geq 0} (f \in \text{Lip}(A, Y, \sigma) \Rightarrow g \in \text{Lip}(X, Y, \sigma))$. Next follows the Lipschitz-version of the extendability of a uniformly continuous function defined on a dense subset of a metric space with values in a complete metric space. Its expected proof, which is based on the proof of Lemma 3.7 in [4], pp. 91-2, is omitted.

Proposition 1.12. Let $D \subseteq X$ be dense in X, Y complete, and $f \in \text{Lip}(D, Y)$.

- $(i) \ \exists_{!g \in \mathbb{F}(X,Y)} \big(g_{|A} = f \ \& \ \forall_{\sigma \geq 0} (f \in \operatorname{Lip}(D,Y,\sigma) \Rightarrow g \in \operatorname{Lip}(X,Y,\sigma)) \big).$
- (ii) If L(f) exists and g is as in (i), then L(g) exists and L(g) = L(f).

(iii) If $L^*(f)$ exists and g is as in (i), then $L^*(g)$ exists and $L^*(g) = L^*(f)$.

2. McShane-Whitney pairs

2.1. **Metric spaces.** Throughout this section X is a set equipped with a metric d, and Y is a set equipped with a metric ρ .

Definition 2.1. Let $A \subseteq X$. We call (X, A) a *McShane-Whitney pair*, or simply an *MW-pair*, if for every $\sigma > 0$ and $g \in \text{Lip}(A, \sigma)$ the functions $g^*, *g : X \to \mathbb{R}$ are well-defined, where for every $x \in X$

$$g^*(x) := \sup\{g(a) - \sigma d(x, a) \mid a \in A\},\$$

$$^*g(x) := \inf\{g(a) + \sigma d(x, a) \mid a \in A\}.$$

Note that (X, A) is a McShane-Whitney pair if and only if for every $g \in \text{Lip}(A, 1)$ the function g^* is well-defined, since we can work with $\frac{1}{\sigma}h$, for an arbitrary $\sigma > 0$ and $h \in \text{Lip}(A, \sigma)$, and use the equality $*g = -[(-g)^*]$ for the extension *g. The classical statement "every pair (X, A) is a McShane-Whitney pair" cannot be accepted constructively.

Remark 2.2. The statement " (\mathbb{R}, A) is a McShane-Whitney pair, for every non-empty $A \subset \mathbb{R}$ " implies PEM.

Proof. If P is a syntactically well-formed proposition, let $A := \{0\} \cup \{x \in \mathbb{R} \mid x = 1 \& P\}$. If $g : A \to \mathbb{R}$ is defined by g(a) := a, for every $a \in A$, then $g \in \text{Lip}(A, 1)$. If x := 2, we suppose that

$$g^*(2) := \sup\{a - |2 - a| \mid a \in A\} \in \mathbb{R}.$$

By the constructive version of the classical trichotomy of reals (Corollary (2.17) in [4], p. 26) we have that $g^*(2) > -\frac{3}{2}$ or $g^*(2) < -1$. In the first case, by the definition of $\sup A$ there is $a \in A$ such that $a - |2 - a| > -\frac{3}{2}$. If a = 0, then $0 - |2 - 0| = -2 < -\frac{3}{2}$, hence there must be $a \in \{x \in \mathbb{R} \mid x = 1 \& P\}$ that satisfies this property. This implies P and $g^*(2) = 0 > -\frac{3}{2}$. If $g^*(2) < -1$, we suppose P, hence $A = \{0, 1\}$. Since then $g^*(2) = 0$, we get the contradiction $0 = g^*(2) < -1$ i.e., we showed $\neg P$.

The next theorem reveals the computational content of the classical McShane-Whitney theorem.

Theorem 2.3 (McShane-Whitney). If (X, A) is an MW-pair and $g \in \text{Lip}(A, \sigma)$, the following hold:

- (i) $g^*, *g \in \text{Lip}(X, \sigma)$.
- (ii) $g^*_{|A} = (^*g)_{|A} = g$.
- $(iii) \ \forall_{f \in \text{Lip}(X,\sigma)} (f_{|A} = g \Rightarrow g^* \le f \le {}^*g).$
- (iv) The pair $(g^*, *g)$ is the unique pair of functions satisfying conditions (i)-(iii).

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Proof. (i) Let $x_1, x_2 \in X$ and $a \in A$. Then $d(x_1, a) \leq d(x_2, a) + d(x_2, x_1)$ and $\sigma d(x_1, a) \leq \sigma d(x_2, a) + \sigma d(x_1, x_2)$, therefore

$$g(a) + \sigma d(x_1, a) \le (g(a) + \sigma d(x_2, a)) + \sigma d(x_1, x_2) \Rightarrow$$

$${}^*g(x_1) \le (g(a) + \sigma d(x_2, a)) + \sigma d(x_1, x_2) \Rightarrow$$

$${}^*g(x_1) - \sigma d(x_1, x_2) \le g(a) + \sigma d(x_2, a) \Rightarrow$$

$${}^*g(x_1) - \sigma d(x_1, x_2) \le {}^*g(x_2) \Rightarrow$$

$${}^*g(x_1) - {}^*g(x_2) \le \sigma d(x_1, x_2).$$

If we start from the inequality $d(x_2, a) \leq d(x_1, a) + d(x_2, x_1)$ and work as above, we get $*g(x_2) - *g(x_1) \leq \sigma d(x_1, x_2)$, therefore $|*g(x_1) - *g(x_2)| \leq \sigma d(x_1, x_2)$. Working similarly we get that g^* is an extension of g which is in $\text{Lip}(X, \sigma)$.

(ii) We show that *g extends g. If $a_0 \in A$, then $*g(a_0) = \inf\{g(a) + \sigma d(a_0, a) \mid a \in A\} \le g(a_0) + \sigma d(a_0, a_0) = g(a_0)$. If $a \in A$, then $g(a_0) - g(a) \le |g(a_0) - g(a)| \le \sigma d(a_0, a)$, hence $g(a) + \sigma d(a_0, a) \ge g(a_0)$. Since a is arbitrary, $*g(a_0) \ge g(a_0)$.

(iii) If $f \in \text{Lip}(X, \sigma)$ such that $f_{|A} = g$, $x \in X$ and $a \in A$ we have that

$$f(x) - g(a) = f(x) - f(a) \le |f(x) - f(a)| \le \sigma d(x, a) \Rightarrow$$

$$f(x) \le g(a) + \sigma d(a, x) \Rightarrow$$

$$f(x) \le *g(x),$$

$$g(a) - f(x) = f(a) - f(x) \le |f(a) - f(x)| \le \sigma d(x, a) \Rightarrow$$

$$g(a) - \sigma d(a, x) \le f(x) \Rightarrow$$

$$g^*(x) \le f(x).$$

(iv) Let h^* , *h satisfy conditions (i)-(iii). Since $h^*_{|A} = (^*h)_{|A} = g$, we have that $g^* \leq h^* \leq ^*g$ and $g^* \leq ^*h \leq ^*g$. Since $g^*_{|A} = (^*g)_{|A} = g$, we have that $h^* \leq g^* \leq ^*h$ and $h^* \leq ^*g \leq ^*h$, hence $h^* = g^*$ and $^*h = ^*g$.

Since in the previous proof we only used the "least upper bound" property of sup and the "greatest lower bound" property of inf, we could have defined g^* and g^* and g^* through lub and glb, respectively. Next we recall some fundamental definitions in the constructive theory of metric spaces.

Definition 2.4. A space X is *compact*, if it is complete and totally bounded, while it is *locally* compact (totally bounded), if every bounded subset of X is included in a compact (totally bounded) subset of X. A subset A of X is *located*, if $d(x, A) := \inf \Delta(x, A)$, where $\Delta(x, A) := \{d(x, a) \mid a \in A\}$, exists, for every $x \in X$.

Proposition 2.5. The following sets X and A form MW-pairs.

- (i) A is a totally bounded subset of X.
- (ii) X is totally bounded and A is located.
- (iii) X is locally compact (totally bounded) and A is bounded and located.

Proof. (i) If $\sigma > 0$, $g \in \text{Lip}(A)$ and $x \in X$, then $g + \sigma d_x$, $g - \sigma d_x \in \text{Lip}(A) \subseteq C_u(A)$, hence $g^*(x)$ and $g^*(x)$ are well defined, since A is totally bounded (see [4], Corollary 4.3, p.94).

- (ii) A located subset of X is also totally bounded (see [4], p.95), and we use (i).
- (iii) If A is bounded and located, there is compact (totally bounded) $K \subseteq X$ with $A \subseteq K$.

Since A is located in X, it is located in K, hence A is totally bounded, and we use (i).

Proposition 2.6. Let (X, A) be an MW-pair and $g \in \text{Lip}(A, \sigma)$. The following hold.

- (i) The set A is located.
- (ii) If $M \in \mathbb{R}$ such that $\forall_{a \in A} (g(a) \leq M)$, then $\forall_{x \in X} (g^*(x) \leq M)$.
- (iii) If $m \in \mathbb{R}$ such that $\forall_{a \in A} (g(a) \geq m)$, then $\forall_{x \in X} (*g(x) \geq m)$.
- (iv) If $\inf g$ and $\sup g$ exist, then $\inf *g$, $\sup g*$ exist and the following equalities hold

$$\inf_{x\in X} {}^*g = \inf_{a\in A} g \quad \& \quad \sup_{x\in X} g^* = \sup_{a\in A} g.$$

Proof. (i) Let $r \in \mathbb{R}$ and $\sigma > 0$. Since $\overline{r}_A \in \text{Lip}(A, \sigma)$, by hypothesis $*\overline{r}_A$ is well-defined, where $*\overline{r}_A(x) = \inf\{r + \sigma d(x, a) \mid a \in A\}$, for every $x \in X$. If $x \in X$ and $a \in A$, then $d(x, a) = \frac{1}{\sigma}(r + \sigma d(x, a) - r)$, and $\Delta(x, A) = \{\frac{1}{\sigma}(r + \sigma d(x, a) - r) \mid a \in A\}$. Hence

$$d(x,A) = \inf \left\{ \frac{1}{\sigma} (r + \sigma d(x,a) - r) \mid a \in A \right\}$$
$$= \frac{1}{\sigma} (\inf \{ r + \sigma d(x,a) \mid a \in A \} - r)$$
$$= \frac{1}{\sigma} (*\overline{r}_A(x) - r).$$

- (ii) Let $x \in X$. Since $g(a) \leq M$, $g(a) \sigma d(x, a) \leq M \sigma d(x, a) \leq M$. Hence $g^*(x) \leq M$.
- (iii) Let $x \in X$. Since $g(a) \ge M$, $g(a) + \sigma d(x, a) \ge M$. Hence $*g \le M$.
- (iv) We show that $m:=\inf\{g(a)\mid a\in A\}$ satisfies the properties of $\inf\{{}^*g(x)\mid x\in X\}$. It suffices to show that $m\leq {}^*g(X)$, since the other definitional condition of inf follows immediately; if $\epsilon>0$, then there exists $a\in A\subseteq X$ such that $g(a)={}^*g(a)< m+\epsilon$. If $x\in A$, then $m\leq g(x)={}^*g(x)$, since $m=\inf g$. Since A is located, the set

$$-A := \{ x \in X \mid d(x, A) > 0 \}$$

is well-defined. If $x \in -A$, then $d(x, a) \geq d(x, A) > 0$, for every $a \in A$. Hence

$$g(a) + \sigma d(x, a) > g(a) \ge \inf_{a \in A} g$$

$$\Rightarrow \inf_{a \in A} (g(a) + \sigma d(x, a)) \ge \inf_{a \in A} g$$

$$\Leftrightarrow {}^*g(x) \ge m.$$

Since A is located, the set $A \cup (-A)$ is dense in X (see [4], p.88). If $x \in X$, there is some sequence $(d_n)_{n \in \mathbb{N}} \subseteq A \cup (-A)$ such that $d_n \stackrel{n}{\to} x$. By the continuity of g we have that $g(d_n) \stackrel{n}{\to} g(x)$. Suppose that g(x) < m. Since $g(d_n) \ge m$, for every $g(d_n) \ge m$, if g(x) > 0, there is some $g(d_n) \ge m$, we have that

$$|*g(d_n) - *g(x)| = *g(d_n) - *g(x) < m - *g(x) \Leftrightarrow *g(d_n) < m,$$

which is a contradiction. Hence ${}^*g(x) \geq m$. For the existence of $\sup g^*$ and the equality $\sup_{x \in X} g^* = \sup_{a \in A} g$ we work similarly.

If X is a normed space and $x_0 \in X$, then it is not generally the case that the set $\mathbb{R}x_0 := \{\lambda x_0 \mid \lambda \in \mathbb{R}\}$ is a located subset of X. If $X = \mathbb{R}$, the locatedness of $\mathbb{R}x_0$, for every $x_0 \in \mathbb{R}$, is equivalent to LPO (see [4], p. 122). If for every $x_0 \in \mathbb{R}$ we have that $(\mathbb{R}, \mathbb{R}x_0)$ is a McShane-Whitney pair, then by Proposition 2.6(i) every set of the form $\mathbb{R}x_0$ is located, which implies LPO.

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The next proposition expresses the "step-invariance" of the McShane-Whitney extension. If $A \subseteq B \subseteq X$ such that (X,A),(X,B) and (B,A) are McShane-Whitney pairs and $g \in \text{Lip}(A)$, then g^{*_X} is the (A-X) extension of $g, g^{*_B*_X}$ is the (B-X) extension of the (A-B) extension g^{*_B} of g, and similarly for ${}^{*_X}g$ and ${}^{*_X*_B}g$.

Proposition 2.7. If $A \subseteq B \subseteq X$ such that (X, A), (X, B), (B, A) are MW-pairs and $g \in \text{Lip}(A, \sigma)$, for some $\sigma > 0$, then

$$g^{*X} = g^{*B*X}, \quad {}^{*X}g = {}^{*X*B}g.$$

Proof. We show only the first equality and for the second we work similarly. By definition $g^{*_B}: B \to \mathbb{R} \in \text{Lip}(B, \sigma)$ and $g^{*_B}(b) = \sup\{g(a) - \sigma d(b, a) \mid a \in A\}$, for every $b \in B$. Moreover, $g^{*_B*_X}: X \to \mathbb{R} \in \text{Lip}(X, \sigma)$ and

$$g^{*_B*_X}(x) = \sup\{g^{*_B}(b) - \sigma d(x, b) \mid b \in B\},\$$

for every $x \in X$. For the (A-X) extension of g we have that $g^{*x}: X \to \mathbb{R} \in \text{Lip}(X, \sigma)$ and $g^{*x}(x) = \sup\{g(a) - \sigma d(x, a) \mid a \in A\}$, for every $x \in X$. Since $(g^{*B*X})_{|B} = g^{*B}$, we have that $(g^{*B*X})_{|A} = (g^{*B})_{|A} = g$. Therefore $g^{*x} \leq g^{*B*X} \leq {}^{*x}g$, and $(g^{*x})_{|B} \leq (g^{*B*X})_{|B} = g^{*B} \leq ({}^{*x}g)_{|B}$. Since $(g^{*x})_{|A} = g$, we get that $((g^{*x})_{|B})_{|A} = g$, therefore $g^{*B} \leq ((g^{*x})_{|B}) \leq {}^{*B}g$. Since $(g^{*x})_{|B} \leq g^{*B}$ and $g^{*B} \leq (g^{*x})_{|B}$, we get $(g^{*x})_{|B} = g^{*B}$. Hence $g^{*B*X} \leq g^{*x} \leq {}^{*x*B}g$ i.e., we have shown both $g^{*x} \leq g^{*B*X}$ and $g^{*B*X} \leq g^{*x}$.

Proposition 2.8. If (X, A) is an MW-pair and $g \in \text{Lip}(A)$ such that L(g) exists, the following hold.

- (i) $g \in \text{Lip}(A, L(g))$.
- (ii) If f is an L(g)-Lipschitz extension of g, then L(f) exists and L(f) = L(g).
- (iii) $L(*g), L(g^*)$ exist and $L(*g) = L(g) = L(g^*)$.

Proof. (i) See our comment before the Proposition 1.11.

- (ii) Since $f \in \text{Lip}(X, L(g))$, we get $L(g) \in \Lambda(f)$ and $L(g) \geq M_0(f)$. Let $\epsilon > 0$. Since $L(g) = \sup M_0(g)$, there exists $(a, b) \in A_0 \subseteq X_0$ such that $\sigma_{a,b}(g) > L(g) \epsilon$. Since f extends g, $\sigma_{a,b}(g) = \sigma_{a,b}(f)$.
- (iii) By definition ${}^*q, q^* \in \operatorname{Lip}(X, L(q))$ and they extend q. Hence we use (ii).

Proposition 2.9. Let (X, A) be an MW-pair, $g_1 \in \text{Lip}(A, \sigma_1), g_2 \in \text{Lip}(A, \sigma_2)$ and $g \in \text{Lip}(A, \sigma)$, for some $\sigma_1, \sigma_2, \sigma > 0$. The following hold.

- (i) $(g_1 + g_2)^* \le g_1^* + g_2^*$ and $(g_1 + g_2) \ge g_1^* + g_2^*$.
- (ii) If $\lambda > 0$, then $(\lambda g)^* = \lambda g^*$ and $(\lambda g) = \lambda^* g$.
- (iii) If $\lambda < 0$, then $(\lambda g)^* = \lambda^* g$ and $(\lambda g)^* = \lambda g^*$.

Proof. In each case we show only one of the two facts.

(i) We have that $g_1 + g_2 \in \text{Lip}(A, \sigma_1 + \sigma_2)$ and

$$(g_1 + g_2)^*(x) = \sup\{g_1(a) + g_2(a) - (\sigma_1 + \sigma_2)d(x, a) \mid a \in A\}$$

$$= \sup\{(g_1(a) - \sigma_1 d(x, a)) + (g_2(a) - \sigma_2 d(x, a)) \mid a \in A\}$$

$$\leq \sup\{g_1(a) - \sigma_1 d(x, a) \mid a \in A\} + \sup\{g_2(a) - \sigma_2 d(x, a) \mid a \in A\}$$

$$= g_1^*(x) + g_2^*(x).$$

(ii) If $\lambda \in \mathbb{R}$, then $\lambda g \in \text{Lip}(A, |\lambda|\sigma)$ and if $\lambda > 0$, then

$$(\lambda g)^*(x) = \sup\{\lambda g(a) - |\lambda|\sigma d(x, a) \mid a \in A\}$$
$$= \lambda \sup\{g(a) - \sigma d(x, a) \mid a \in A\}$$
$$= \lambda g^*(x).$$

(iii) If $\lambda < 0$, then

$$(\lambda g)^*(x) = \sup\{\lambda g(a) - |\lambda|\sigma d(x, a) \mid a \in A\}$$

$$= \sup\{\lambda g(a) - (-\lambda)\sigma d(x, a) \mid a \in A\}$$

$$= \sup\{\lambda (g(a) + \sigma d(x, a)) \mid a \in A\}$$

$$= \lambda \inf\{g(a) + \sigma d(x, a) \mid a \in A\}$$

$$= \lambda^* g(x).$$

2.2. Local McShane-Whitney pairs. Next we define a local notion of an MW-pair.

Definition 2.10. The pair (X, A) is a *local* McShane-Whitney pair, or simply a local MW-pair, if for every bounded $B \subseteq A$ there is $A' \subseteq A$ such that $B \subseteq A'$ and (X, A') is a MW-pair.

Clearly, an MW-pair is a local MW-pair. The next lemma is shown in [6], p. 33 and in [7], p. 46. If $A, B \subseteq X$, we use Sambin's notation $A \not \setminus B$ to denote that the intersection $A \cap B$ is inhabited.

Lemma 2.11. If A is a located subset of a metric space X and T is a totally bounded subset of X such that $T \ \)$ A, there is totally bounded $S \subseteq X$ such that $T \cap A \subseteq S \subseteq A$.

The next proposition is the "local" version of Proposition 2.5(i)-(ii).

Proposition 2.12. If $A \subseteq X$, the following hold.

- (i) If X is locally totally bounded and A is located, then (X, A) is a local MW-pair.
- (ii) If A is a locally totally bounded subset of X, then (X, A) is a local MW-pair.
- *Proof.* (i) Let $B \subseteq A$ bounded. By the definition of local total boundedness there is a totally bounded $T \subseteq X$ such that $B \subseteq T$. Since $B \subseteq T \cap A$, and since by the definition of a bounded set, B is inhabited, we get $A \not \setminus T$. By Lemma 2.11 there is a totally bounded $S \subseteq X$ such that $B \subseteq T \cap A \subseteq S \subseteq A$. Since S is totally bounded, by Proposition 2.5(i) we have that (X, S) is an MW-pair.
- (ii) Let $B \subseteq A$ be bounded. By the definition of local total boundedness there is a totally bounded $T \subseteq A$ such that $B \subseteq T$. As in case (i), we get that (X, T) is an MW-pair.
- 2.3. Hölder continuous functions of order α and ν -continuous functions. As expected, the McShane-Whitney extension works for the standard generalizations of Lipschitz functions.

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Definition 2.13. If $\sigma \geq 0$ and $\alpha \in (0,1]$, the Hölder continuous functions of order α are defined as follows:

$$\begin{split} \mathrm{H\"ol}(X,Y,\alpha,\sigma) := \{ f \in \mathbb{F}(X,Y) \mid \forall_{x,y \in X} (\rho(f(x),f(y)) \leq \sigma d(x,y)^{\alpha}) \}, \\ \mathrm{H\"ol}(X,Y,\alpha) := \bigcup_{\sigma \geq 0} \mathrm{H\"ol}(X,Y,\alpha,\sigma). \end{split}$$

If $Y = \mathbb{R}$ is equipped with its standard metric, we write $H\ddot{o}l(X, \alpha, \sigma)$ and $H\ddot{o}l(X, \alpha)$, respectively.

Clearly, $H\ddot{o}l(X,Y,1,\sigma) = \text{Lip}(X,Y,\sigma)$ and $H\ddot{o}l(X,Y,1) = \text{Lip}(X,Y)$. One can define MW-pairs with respect to Hölder continuous functions of order α , and show a McShane-Whitney extension theorem for them. We give some more details of these two steps only for the following generalization of Hölder continuity. The next definition is a reformulation of the definition found in the formulation of Theorem 5.6 in [4], p. 102.

Definition 2.14. A modulus of continuity is a function $\nu : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

- (i) $\nu(0) = 0$.
- (ii) $\forall_{x,y \in [0,+\infty)} (\nu(x+y) \le \nu(x) + \nu(y)).$
- (iii) It is strictly increasing i.e., $\forall_{s,t \in [0,+\infty)} (s < t \Rightarrow \nu(s) < \nu(t))$.
- (iv) It is uniformly continuous on every bounded subset of $[0, +\infty)$.

If ν is a modulus of continuity the set of ν -continuous functions from X to Y, and from X to \mathbb{R} , are defined, respectively, as follows:

$$S(X,Y,\nu) := \{ f \in C_u(X,Y) \mid \forall_{x,y \in X} (\rho(f(x), f(y)) \le \nu(d(x,y)) \},$$

$$S(X,\nu) := \{ f \in C_u(X) \mid \forall_{x,y \in X} (|f(x) - f(y)| \le \nu(d(x,y)) \},$$

Clearly, if $\nu_1(t) = \sigma t$, $\nu_2(t) = \sigma t^{\alpha}$, for every $t \in [0, +\infty)$, then

$$S(X, \nu_1) = \operatorname{Lip}(X, \sigma),$$

$$S(X, \nu_2) = \text{H\"ol}(X, \alpha, \sigma).$$

Notice that since $a \in (0,1]$, we have that ν_2 is concave, hence subadditive. It is immediate to show that if ν is a modulus of continuity, then

$$\forall_{s,t\in[0,+\infty)} (s \le t \Rightarrow \nu(s) \le \nu(t)).$$

Definition 2.15. Let A be a subset of X. We call (X, A) a ν -McShane-Whitney pair, or simply a ν -MW-pair, if for every $g \in S(A, \nu)$ the functions $g_{\nu}^*, {}^*g_{\nu}: X \to \mathbb{R}$ are well-defined, where for every $x \in X$

$$g_{\nu}^{*}(x) := \sup\{g(a) - \nu(d(x, a)) \mid a \in A\},$$

$${}^{*}g_{\nu}(x) := \inf\{g(a) + \nu(d(x, a)) \mid a \in A\}.$$

In order to get examples of ν -MW-pairs, one proceeds as in Proposition 2.5(i)-(iii). If, for example, A is totally bounded, then, since d_x is uniformly continuous, $d_x(A)$ is a bounded subset of $[0, +\infty)$ and $\nu \circ d_x = \nu_{|d_x(A)} \circ d_x$ is a uniformly continuous function as a composition of uniformly continuous functions. Hence ${}^*g_{\nu}(x)$ and $g_{\nu}^*(x)$ are well-defined.

Theorem 2.16 (McShane-Whitney theorem for ν -continuous functions). If (X, A) is a ν -MW-pair and $g \in S(A, \nu)$, the following hold.

- (i) $g_{\nu}^*, {}^*g_{\nu} \in S(X, \lambda)$.
- $(ii) g_{\nu|A}^* = (*g_{\nu})_{|A} = g.$
- $(iii) \ \forall_{f \in S(X,\nu)} (f_{|A} = g \Rightarrow g_{\nu}^* \le f \le {}^*g_{\nu}).$
- (iv) The pair $(g_{\nu}^*, *g_{\nu})$ is the unique pair of functions satisfying conditions (i)-(iii).

Proof. Based on the properties of Definition 2.14, all steps of the proof of Theorem 2.3 are repeated for the functions $*g_{\nu}, g_{\nu}^*$. To show the implication $d(x_1, a) \leq d(x_2, a) + d(x_1, x_2) \Rightarrow \nu(d(x_1, a)) \leq \nu(d(x_2, a) + d(x_1, x_2)) \leq \nu(d(x_2, a)) + \nu(d(x_1, x_2))$ we use the fact that $\forall_{s,t \in [0,+\infty)} (s \leq t \Rightarrow \nu(s) \leq \nu(t))$.

3. Normed spaces

3.1. McShane-Whitney pairs in normed spaces. Throughout this section X is a normed space with norm ||.||.

Definition 3.1. A subset C of X is called convex, if $\forall_{x,y\in C}\forall_{\lambda\in(0,1)}(\lambda x+(1-\lambda)y\in C)$. If $C\subseteq X$ is convex, a function $g:C\to\mathbb{R}$ is called convex, if $\forall_{x,y\in C}\forall_{\lambda\in[0,1]}(g(\lambda x+(1-\lambda)y)\leq \lambda g(x)+(1-\lambda)g(y))$, and g is called concave, if $\forall_{x,y\in C}\forall_{\lambda\in[0,1]}(g(\lambda x+(1-\lambda)y)\geq \lambda g(x)+(1-\lambda)g(y))$. A function $f:X\to\mathbb{R}$ is called sublinear if it is subadditive and positively homogeneous i.e., if $f(x+y)\leq f(x)+f(y)$, and $f(\lambda x)=\lambda f(x)$, for every $x,y\in X$ and $\lambda>0$, respectively. Similarly, f is called superlinear, if it is superadditive i.e., if $f(x+y)\geq f(x)+f(y)$, for every $x,y\in X$, and positively homogeneous.

Proposition 3.2. If $C \subseteq X$ is convex, (X,C) is an MW-pair, and $g \in \text{Lip}(C,\sigma)$, the following hold.

- (i) If g is convex, then *g is convex.
- (ii) If g is concave, then g^* is concave.

Proof. We show only (i), and for (ii) we work similarly. Let $x, y \in X$, and $\lambda \in [0, 1]$. If

$$C_x = \{ \lambda(g(c) + \sigma || x - c ||) \mid c \in C \},$$

$$D_y = \{ (1 - \lambda)(g(c) + \sigma || y - c ||) \mid c \in C \},$$

an element of $C_x + D_y$ has the form $\lambda(g(c) + \sigma||x - c||) + (1 - \lambda)(g(d) + \sigma||y - d||)$, for some $c, d \in C$. We show that

$$*g(\lambda x + (1 - \lambda)y) \le \lambda(g(c) + \sigma||x - c||) + (1 - \lambda)(g(d) + \sigma||y - d||),$$

where $c, d \in C$. Since C is convex, $c' := \lambda c + (1 - \lambda)d \in C$, and by the convexity of q we get

$$*g(\lambda x + (1 - \lambda)y) \le g(c') + \sigma ||\lambda x + (1 - \lambda)y - c'||$$

$$\le \lambda g(c) + (1 - \lambda)g(d) + \lambda \sigma ||x - c|| + (1 - \lambda)\sigma ||y - d||$$

$$= \lambda (g(c) + \sigma ||x - c||) + (1 - \lambda)(g(d) + \sigma ||y - d||).$$

Since the element of $C_x + D_y$ considered is arbitrary, we get that ${}^*g(\lambda x + (1 - \lambda)y) \le \inf(C_x + D_y) = \inf C_x + \inf D_y = \lambda^*g(x) + (1 - \lambda)^*g(y)$.

The next proposition says that if g is linear, then *g is sublinear and g^* is superlinear.

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Proposition 3.3. If A is a non-trivial subspace of X such that (X, A) is an MW-pair, and if $g \in \text{Lip}(A, \sigma)$ is linear, the following hold.

(i)
$$g^*(x_1 + x_2) \ge g^*(x_1) + g^*(x_2)$$
, and $g^*(x_1 + x_2) \le g^*(x_1) + g^*(x_2)$.

(ii) If
$$\lambda > 0$$
, then $q^*(\lambda x) = \lambda q^*(x)$ and $q(\lambda x) = \lambda^* q(x)$.

(iii) If
$$\lambda < 0$$
, then $g^*(\lambda x) = \lambda^* g(x)$ and $g^*(\lambda x) = \lambda g^*(x)$.

Proof. In each case we show only one of the two facts.

(i) If $a_1, a_2 \in A$, then

$$*g(x_1 + x_2) = \inf\{g(a) + \sigma | |(x_1 + x_2) - a| | | a \in A\}$$

$$\leq g(a_1 + a_2) + \sigma | |x_1 + x_2 - (a_1 + a_2)| |$$

$$\leq g(a_1) + \sigma | |x_1 - a_1| + g(a_2) + \sigma | |x_2 - a_2| |,$$

therefore

$$*g(x_1 + x_2) \le \inf\{g(a_1) + \sigma | |x_1 - a_1|| + g(a_2) + \sigma | |x_2 - a_2|| \mid a_1, a_2 \in A\}$$

$$= \inf\{g(a_1) + \sigma | |x_1 - a_1|| \mid a_1 \in A\} + \inf\{g(a_2) + \sigma | |x_2 - a_2|| \mid a_2 \in A\}$$

$$= *g(x_1) + *g(x_2).$$

(ii) First we show that ${}^*g(\lambda x) \leq \lambda^*g(x)$. If $a \in A$, then

$$*g(\lambda x) = \inf\{g(a) + \sigma ||\lambda x - a|| \mid a \in A\}$$

$$\leq g(\lambda a) + \sigma ||\lambda x - \lambda a||$$

$$= \lambda g(a) + |\lambda |\sigma||x - a||$$

$$= \lambda (g(a) + \sigma ||x - a||),$$

therefore

$${}^*g(\lambda x) \le \inf\{\lambda(g(a) + \sigma||x - a||) \mid a \in A\}$$
$$= \lambda \inf\{g(a) + \sigma||x - a|| \mid a \in A\}$$
$$= \lambda^*g(x).$$

For the inequality ${}^*g(\lambda x) \ge \lambda^*g(x)$ we work as follows.

$$\lambda^* g(x) = \lambda \inf \{ g(a) + \sigma || x - a|| \mid a \in A \}$$

$$\leq \lambda (g(\frac{1}{\lambda}a) + \sigma || x - \frac{1}{\lambda}a||)$$

$$= g(a) + \sigma |\lambda|||x - \frac{1}{\lambda}a||$$

$$= g(a) + \sigma ||\lambda(x - \frac{1}{\lambda}a)||$$

$$= g(a) + \sigma ||\lambda x - a||,$$

therefore

$$\lambda^* g(x) \le \inf\{g(a) + \sigma ||\lambda x - a|| \mid a \in A\} = {}^*g(\lambda x).$$

(iii) First we show that $*g(\lambda x) \leq \lambda g^*(x)$. If $a \in A$, then

$$*g(\lambda x) = \inf\{g(a) + \sigma ||\lambda x - a|| \mid a \in A\}$$

$$\leq g(\lambda a) + \sigma ||\lambda x - \lambda a||$$

$$= \lambda g(a) + |\lambda |\sigma||x - a||$$

$$= \lambda (g(a) - \sigma ||x - a||),$$

therefore

$${}^*g(\lambda x) \le \inf\{\lambda(g(a) - \sigma||x - a||) \mid a \in A\}$$
$$= \lambda \sup\{g(a) - \sigma||x - a|| \mid a \in A\}$$
$$= \lambda g^*(x).$$

For the inequality ${}^*g(\lambda x) \geq \lambda g^*(x)$ we work as follows. Since

$$g^*(x) = \sup\{g(a) - \sigma | |x - a|| \mid a \in A\}$$
$$\geq g(\frac{1}{\lambda}a) - \sigma | |x - \frac{1}{\lambda}a||$$

and $\lambda < 0$, we get

$$\lambda g^*(x) \le \lambda (g(\frac{1}{\lambda}a) - \sigma || x - \frac{1}{\lambda}a ||)$$

$$= g(a) - \lambda \sigma || x - \frac{1}{\lambda}a ||$$

$$= g(a) + \sigma |\lambda| || x - \frac{1}{\lambda}a ||$$

$$= g(a) + \sigma || \lambda (x - \frac{1}{\lambda}a) ||$$

$$= g(a) + \sigma || \lambda x - a ||,$$

therefore

$$\lambda q^*(x) < \inf\{q(a) + \sigma || \lambda x - a|| \mid a \in A\} = {}^*q(\lambda x).$$

As we have already said in subsection 2.1, if $x_0 \in X$, it is not generally the case that $\mathbb{R}x_0 := \{\lambda x_0 \mid \lambda \in \mathbb{R}\}$ is a located subset of X. If $||x_0|| > 0$ though, $\mathbb{R}x_0$ is a 1-dimensional subspace of X i.e., a closed and located linear subset of X of dimension one (see [4], p. 307). Of course, $\mathbb{R}x_0$ is a convex subset of X.

A standard corollary of the classical Hahn-Banach theorem is that if $x_0 \neq 0$, there is a bounded linear functional u on X such that ||u|| = 1 and $u(x_0) = ||x_0||$. Its proof is based on the extension of the obvious linear map on $\mathbb{R}x_0$ to X through the Hahn-Banach theorem. Next follows a first approach to the translation of this corollary in Lipschitz analysis. Of course, if $x_0 \in X$, then $||.|| \in \text{Lip}(X)$ and L(||.||) = 1. To find a Lipschitz function on X, which is also the McShane-Whitney extension of some Lipschitz function defined on the set of a non-trivial McShane-Whitney pair, we work as follows. It is immediate to see that if $x_0 \in X$ such that $||x_0|| > 0$, then $||x_0|| = ||\lambda x_0|| ||\lambda \in [-1, 1]|$ is a compact subset of X. If we define $g : ||x_0|| \to ||x_0||$ by $g(\lambda x_0) = \lambda ||x_0||$, for every $\lambda \in [-1, 1]$, then $g \in \text{Lip}(||x_0||)$ and L(g) = 1; if $\lambda, \mu \in [-1, 1]$, then $||g(\lambda x_0) - g(\mu x_0)|| = ||\lambda|||x_0|| - \mu ||x_0||| = ||\lambda - \mu|||x_0||| = ||(\lambda - \mu)x_0|| = ||\lambda x_0 - \mu x_0||$, and since

$$M_0(g) = \left\{ \sigma_{\lambda x_0, \mu x_0}(g) = \frac{|g(\lambda x_0) - g(\mu x_0)|}{||\lambda x_0 - \mu x_0||} = 1 \mid (\lambda, \mu) \in [-1, 1]_0 \right\},\,$$

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we get that $L(g) = \sup M_0(g) = 1$. Since $\mathbb{I}x_0$ is totally bounded, by Proposition 2.5(i) we have that $(X, \mathbb{I}x_0)$ is an MW-pair, hence the extension *g of g is in $\operatorname{Lip}(X)$. By Proposition 2.8 we have that L(*g) = L(g) = 1.

Theorem 3.4. Let $x_0 \in X$ such that $||x_0|| > 0$. If $(X, \mathbb{R}x_0)$ is a MW-pair, there exists a linear function $g \in \text{Lip}(\mathbb{R}x_0)$ with L(g) = 1, such that *g is a sublinear Lipschitz function on X with $*g(x_0) = ||x_0||$ and L(*g) = 1, and g^* is a superlinear Lipschitz function on X with $g^*(x_0) = ||x_0||$ and $L(g^*) = 1$.

Proof. As in the proof of the previous remark, the function $g : \mathbb{R}x_0 \to \mathbb{R}$, defined by $g(\lambda x_0) = \lambda ||x_0||$, for every $\lambda \in \mathbb{R}$, is in $\operatorname{Lip}(\mathbb{R}x_0)$ and L(g) = 1. Since $(X, \mathbb{R}x_0)$ is an MW-pair, the extension *g of g is a Lipschitz function, and by Proposition 2.8 we get L(*g) = L(g) = 1. Since g is linear, by Proposition 3.3 we have that *g is sublinear. Similarly, the extension g^* of g is a Lipschitz function, and by Proposition 2.8 we get $L(g^*) = L(g) = 1$. Since g is linear, by Proposition 3.3 we have that g^* is superlinear. \square

3.2. Approximate McShane-Whitney theorem for normed spaces. Next we show the approximate version of the McShane-Whitney theorem for normed spaces. The technique of our proof⁸ is found in the proof of the lemma that is necessary to prove the approximate version of the constructive Hahn-Banach theorem (Lemma 5.9 in [6], p. 39). What we get in the conclusion of the approximate McShane-Whitney theorem is not a σ -Lipschitz extension of a given σ -Lipschitz function g, but a $(1+\epsilon)\sigma$ -Lipschitz extension of g, for every $\epsilon>0$, where the Lipschitz function g is defined on a finite-dimensional subspace of a normed space. Of course, if we consider the trivial subspace $\{0\}$ of X, the McShane-Whitney theorem holds trivially.

Theorem 3.5 (Approximate McShane-Whitney theorem for normed spaces). Let A be a non-trivial, finite dimensional, linear subspace of X and $g \in \text{Lip}(A, \sigma)$, for some $\sigma > 0$. If $\epsilon > 0$, there exist functions g_{ϵ}^* , ${}^*g_{\epsilon}: X \to \mathbb{R}$ such that:

- (i) $g_{\epsilon}^*, *g_{\epsilon} \in \text{Lip}(X, (1+\epsilon)\sigma)$.
- $(ii) (g_{\epsilon}^*)_{|A} = (*g_{\epsilon})_{|A} = g.$
- (iii) $\forall_{f \in \text{Lip}(X,(1+\epsilon)\sigma)} (f_{|A} = g \Rightarrow g_{\epsilon}^* \le f \le {}^*g_{\epsilon}).$
- (iv) The pair $(g_{\epsilon}^*, *g_{\epsilon})$ is the unique pair of functions satisfying conditions (i)-(iii).

Proof. Suppose first that q(0) = 0.

Let $a, b \in A$ and $x \in X$. Since $g(a) - g(b) \le \sigma ||a - b|| \le \sigma ||a - x|| + \sigma ||x - b||$, we have that $g(a) - \sigma ||a - x|| \le g(b) + \sigma ||x - b||$,

hence

$$\phi_{\epsilon,x}(a) := g(a) - (1+\epsilon)\sigma||a-x|| \le g(b) + (1+\epsilon)\sigma||x-b|| =: \theta_{\epsilon,x}(b).$$

We find r > 0, such that for every $a \in A$

$$||a|| \ge r \Rightarrow \phi_{\epsilon,x}(a) \le -(1+\epsilon)\sigma||x|| = \phi_{\epsilon,x}(0). \tag{3.1}$$

Let

$$r := \frac{2(1+\epsilon)}{\epsilon}||x||.$$

⁸The author would like to thank Hajime Ishihara for bringing this technique to our attention.

We show first that

$$||a|| \ge r \Rightarrow ||a|| - (1+\epsilon)||a-x|| \le -(1+\epsilon)||x||.$$
 (3.2)

We have that

$$\begin{split} ||a-x|| &\geq ||a|| - ||x|| \Rightarrow -||a-x|| \leq ||x|| - ||a|| \\ &\stackrel{\times (1+\epsilon)}{\Rightarrow} -(1+\epsilon)||a-x|| \leq (1+\epsilon)||x|| - (1+\epsilon)||a|| \\ &\stackrel{+||a||}{\Rightarrow} ||a|| - (1+\epsilon)||a-x|| \leq (1+\epsilon)||x|| - (1+\epsilon)||a|| + ||a|| \\ &\Leftrightarrow ||a|| - (1+\epsilon)||a-x|| \leq (1+\epsilon)||x|| - \epsilon||a||. \end{split}$$

By the definition of r we have that

$$||a|| \ge \frac{2(1+\epsilon)}{\epsilon}||x|| \Rightarrow -||a|| \le -\frac{2(1+\epsilon)}{\epsilon}||x||$$

$$\stackrel{\times \epsilon}{=} -\epsilon||a|| \le -2(1+\epsilon)||x||,$$

hence

$$||a|| - (1+\epsilon)||a - x|| \le (1+\epsilon)||x|| - \epsilon||a|| \le (1+\epsilon)||x|| - 2(1+\epsilon)||x|| = -(1+\epsilon)||x||.$$

Hence, if $||a|| \ge r$, multiplying the conclusion of (3.2) by $\sigma > 0$ we get

$$\sigma||a|| - (1+\epsilon)\sigma||a-x|| \le -(1+\epsilon)\sigma||x||.$$

Since

$$g(a) = g(a) - g(0) \le |g(a) - g(0)| \le \sigma ||a - 0|| = \sigma ||a||,$$

we get

$$g(a) - (1 + \epsilon)\sigma||a - x|| \le -(1 + \epsilon)\sigma||x||,$$

which is the required conclusion of (3.1). Since A is finite dimensional space, the set $K_r := \{a \in A \mid ||a|| \le r\}$ is compact (see [6], p. 34), and

$$s_{\epsilon,x} := \sup \{ \phi_{\epsilon,x}(a) \mid a \in K_r \}$$

is well defined, since ϕ is uniformly continuous and K_r is totally bounded. Since $0 \in K_r$ and since the set $\{\phi_{\epsilon,x}(a) \mid a \in A \& ||a|| \ge r\}$ is bounded above by $\phi_{\epsilon,x}(0)$, we get⁹

$$s_{\epsilon,x} = \sup \{ \phi_{\epsilon,x}(a) \mid a \in A \}.$$

Hence we define

$$q_{\epsilon}^*(x) := s_{\epsilon,x} = \sup\{q(a) - (1+\epsilon)\sigma | |x-a| | | a \in A\}.$$

Next we show that $g_{\epsilon}^* \in \text{Lip}(X, (1+\epsilon)\sigma)$. Since $||x-a|| \leq ||x-y|| + ||y-a||$, we get $g(a) - (1+\epsilon)\sigma||x-a|| \geq g(a) - (1+\epsilon)\sigma||y-a|| - (1+\epsilon)\sigma||x-y||$, and $(1+\epsilon)\sigma||x-y|| + g(a) - (1+\epsilon)\sigma||x-a|| \geq g(a) - (1+\epsilon)\sigma||y-a||$, hence $(1+\epsilon)\sigma||x-y|| + g(a) - (1+\epsilon)\sigma||x-a|| \geq g_{\epsilon}^*(y)$, and $(1+\epsilon)\sigma||x-y|| + g_{\epsilon}^*(x) \geq g_{\epsilon}^*(y)$. Consequently,

$$q_{\epsilon}^{*}(y) - q_{\epsilon}^{*}(x) < (1 + \epsilon)\sigma||x - y||.$$

Working similarly, we also get $g_{\epsilon}^*(x) - g_{\epsilon}^*(y) \le (1+\epsilon)\sigma||x-y||$. Suppose next that $a_0 \in A$. By definition we have that $g_{\epsilon}^*(a_0) = \sup\{g(a) - (1+\epsilon)||a_0 - a|| \mid a \in A\}$. Clearly, $g(a_0) = g(a_0) - (1+\epsilon)\sigma||a_0 - a_0|| \le g_{\epsilon}^*(a_0)$. If $a \in A$, then $g(a) - g(a_0) \le \sigma||a - a_0|| \le (1+\epsilon)\sigma||a - a_0||$, hence $-g(a_0) \le -g(a) + (1+\epsilon)\sigma||a - a_0||$ and $g(a_0) \ge g(a) - (1+\epsilon)\sigma||a - a_0||$. Since $a \in A$

⁹Here we use the simple fact that if $D \subseteq B \subseteq \mathbb{R}$ such that D is dense in B and $\sup D$ exists, then $\sup B$ exists and $\sup B = \sup D$. The set $E = K_r \cup \{a \in A \mid ||a|| \ge r\}$ is dense in A and $\phi_{\epsilon,x}(E)$ is dense in $\phi_{\epsilon,x}(A)$.

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is arbitrary, we get $g(a_0) \geq g_{\epsilon}^*(a_0)$. If $f \in \text{Lip}(X, (1+\epsilon)\sigma)$ such that $f_{|A} = g$, then $g(a) - f(x) = f(a) - f(x) \leq (1+\epsilon)\sigma||x-a||$, hence $g(a) - (1+\epsilon)\sigma||x-a|| \leq f(x)$, and since $a \in A$ is arbitrary, we get $g_{\epsilon}^*(x) \leq f(x)$.

For the construction of ${}^*g_{\epsilon}$ we work similarly, as follows. The same r > 0 that we chose earlier satisfies, for every $a \in A$, also the property

$$||a|| \ge r \Rightarrow \theta_{\epsilon,x}(a) = g(a) + (1+\epsilon)\sigma||x-a|| \ge (1+\epsilon)\sigma||x|| = \theta_{\epsilon,x}(0).$$

The reason for this is that since we can choose $||a|| \ge r$ such that $||a|| \le (1+\epsilon)(||a-x||-||x||)$, we have that

$$\begin{split} &-||a|| \geq (1+\epsilon)(||x|| - ||a-x||) \Rightarrow \\ &-||a|| + (1+\epsilon)(||a-x||) \geq (1+\epsilon)||x|| \Rightarrow \\ &-\sigma||a|| + (1+\epsilon)\sigma(||a-x||) \geq (1+\epsilon)\sigma||x|| \Rightarrow \\ &g(a) + (1+\epsilon)\sigma(||a-x||) \geq (1+\epsilon)\sigma||x||, \end{split}$$

since from the inequality $-g(a) \le |g(a)| \le \sigma |a|$, we get $g(a) \ge -\sigma |a|$. Similarly,

$$\iota_{\epsilon,x} := \inf\{\theta_{\epsilon,x}(a) \mid a \in K_r\}$$

is well-defined. Since the set $\{\theta_{\epsilon,x}(a) \mid a \in A \& ||a|| \ge r\}$ is bounded below by $\theta_{\epsilon,x}(0)$, we get $\iota_{\epsilon,x} = \inf\{\theta_{\epsilon,x}(a) \mid a \in A\}.$

Hence, we define

$$*g_{\epsilon}(x) := \iota_{\epsilon,x} = \inf\{g(a) + (1+\epsilon)\sigma | |x-a| \mid |a \in A\}.$$

To show the properties of ${}^*g_{\epsilon}$ in (i)-(iii), we proceed as we did in the proof of the properties of g_{ϵ}^* . The uniqueness of the pair $(g_{\epsilon}^*, {}^*g_{\epsilon})$ follows immediately.

In the general case, where we do not assume that g(0) = 0, we work as follows. If $h : A \to \mathbb{R}$ is defined by

$$h(a) := g(a) - g(0),$$

for every $a \in A$, then h(0) = 0. Clearly, $h \in \text{Lip}(A, \sigma)$, hence by our proof above there are functions h_{ϵ}^* , $h_{\epsilon}^* : X \to \mathbb{R}$ such that:

- (i)' $h_{\epsilon}^*, *h_{\epsilon} \in \text{Lip}(X, (1+\epsilon)\sigma).$
- $(ii)' (h_{\epsilon}^*)_{|A} = (*h_{\epsilon})_{|A} = h.$
- $(iii)' \forall_{F \in \text{Lip}(X,(1+\epsilon)\sigma)} (F_{|A} = h \Rightarrow h_{\epsilon}^* \leq F \leq {}^*h_{\epsilon}).$
- (iv) The pair $(h_{\epsilon}^*, h_{\epsilon})$ is the unique pair of functions satisfying conditions (i) -(iii).

We define $g_{\epsilon}^*, {}^*g_{\epsilon}: X \to \mathbb{R}$ by

$$g_{\epsilon}^*(x) := h_{\epsilon}^*(x) + g(0),$$

 $g_{\epsilon}^*(x) := h_{\epsilon}^*(x) + g(0),$

for every $x \in X$. Properties (i)-(iv) for g_{ϵ}^* and g_{ϵ}^* follow immediately from properties (i)'-(iv)' for h_{ϵ}^* and h_{ϵ}^* .

4. Concluding remarks and open questions

In this paper we examined the McShane-Whitney extension of Lipschitz functions from the Bishop-style, constructive point of view. As in the case of Tietze's theorem, or the Hahn-Banach theorem, the generality of the classical McShane-Whitney theorem is lost. In reward though, the computational content of this extension is constructively studied through the notion of a McShane-Whitney pair.

Moreover, the analogies between the Hahn-Banach theorem and the McShane-Whitney theorem made possible to transfer methods and results from one to the other. A significant example of the interplay between the two theorems is the approximate McShane-Whitney theorem, the proof of which uses the proof-technique of the approximate Hahn-Banach theorem.

Another example that we plan to elaborate in future work is the following. If p is a seminorm on a vector space X i.e., $p: X \to \mathbb{R}$ such that (i) $p(x) \geq 0$, (ii) $p(x+y) \leq p(x) + p(y)$, and (iii) $p(\lambda x) = |\lambda| p(x)$, for every $x, y \in X$ and $\lambda \in \mathbb{R}$, the set of p-Lipschitz functions on X is defined by

$$\begin{split} p - \mathrm{Lip}(X) &:= \bigcup_{\sigma \geq 0} p - \mathrm{Lip}(X, \sigma), \\ p - \mathrm{Lip}(X, \sigma) &:= \{ f \in \mathbb{F}(X, Y) \mid \forall_{x, y \in X} \big(|f(x) - f(y)| \leq \sigma p(x - y) \big) \}. \end{split}$$

One can define a p-MW-pair in the obvious way. In order to get examples of p-MW-pairs, one can define p-versions of locatedness, total boundedness and uniform continuity, and prove, similarly, the p-versions of the propositions in Bishop and Bridges pp. 94-95 and the p-version of Proposition 2.5. The proof of the expected McShane-Whitney theorem for p-Lipschitz functions is similar to the proof of Theorem 2.3. The following MW-version of the analytic Hahn-Banach theorem for seminorms¹⁰ can be shown.

If p is a seminorm on X, (X, A) is a p-MW-pair, A is a subspace of X, and $g: A \to \mathbb{R}$ is linear, then if $\forall_{a \in A}(g(a) \leq p(a))$, there is a superlinear extension g^* of g such that $g^* \in \text{p-Lip}(X,1)$ and $\forall_{x \in X}(g^*(x) \leq p(x))$, while if $\forall_{a \in A}(g(a) \geq p(a))$, there is a sublinear extension *g of g such that $*g \in \text{p-Lip}(X,1)$ and $\forall_{x \in X}(*g(x) \geq p(x))$.

With respect to the analytic Hahn-Banach theorem for seminorms we lose linearity of the extension function, since in the case (i) of the previous result g^* is only superlinear, but instead we have that $g^* \in \text{p-Lip}(X,1)$, and we avoid Zorn's lemma. We expect to find good applications of the previous result in convex analysis, where linearity is not as important as sublinearity, or superlinearity.

Moreover, the following problems and questions arise naturally from our work.

- (1) To find necessary and sufficient conditions on metric spaces X, Y and on function $f \in \text{Lip}(X,Y)$ such that L(f) and/or $L^*(f)$ exist. This problem is similar to the normability of a bounded linear functional T, which, constructively is equivalent to the locatedness of the kernel of T.
- (2) To characterise, for given metric space X, its MW-pairs (X, A).

 $^{^{10}}$ According to it, if p is a seminorm on X, A a subspace of X and $g: A \to \mathbb{R}$ a linear function such that $\forall_{a \in A} (g(a) \leq p(a))$, there is linear function $f: X \to \mathbb{R}$ that extends g and $\forall_{x \in X} (f(x) \leq p(x))$. In its general formulation, the analytic Hahn-Banach theorem refers to sublinear functionals p, where the condition (iii) of a seminorm is replaced by the condition $p(\lambda x) = \lambda p(x)$, for $\lambda \geq 0$ (see [16], p. 120). Its proof uses Zorn's lemma.

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- (3) To find necessary and/or sufficient conditions on a normed space (X, ||.||) such that $(X, \mathbb{R}x_0)$ is an MW-pair for some given $x_0 \in X$ with $||x_0|| > 0$. We expect that the properties added by Ishihara in [13] to the hypotheses of the Hahn-Banach theorem i.e., the uniform convexity and separability of X and the Gâteaux differentiability of its norm, are interesting properties to be tested for this question too.
- (4) To connect the separation version of the Hahn-Banach theorem to the McShane-Whitney theorem.

We hope to address these questions and problems in future work.

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References

- [1] P. Aczel, M. Rathjen: Constructive Set Theory, book draft, 2010.
- [2] Y. Benyamini and J. Lindenstrauss: Geometric nonlinear functional analysis, Vol.1, American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000.
- [3] E. Bishop: Foundations of Constructive Analysis, McGraw-Hill, 1967.
- [4] E. Bishop and D. Bridges: Constructive Analysis, Grundlehren der Math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.
- [5] A. Brudnyi and Y. Brudnyi: Methods of geometric analysis in extension and trace problems, Volume 1, Monographs in Mathematics, 102, Birkhäuser/Springer Basel AG, Basel, 2012.
- [6] D. S. Bridges and F. Richman: Varieties of Constructive Mathematics, Cambridge University Press, 1987.
- [7] D. S. Bridges and L. S. Vîţă: Techniques of Constructive Analysis, in: Universitext, Springer, New York, 2006.
- [8] D. Bridges, M. Hendtlass, E. Palmgren: A Constructive Examination of Rectifiability, Journal of Logic and Analysis, Vol. 8, 2016.
- [9] J. Conway: A Course in Abstract Analysis, American Mathematical Society, 2012.
- [10] O. Demuth: On the differentiability of constructive functions, Comment. Math. Univ. Carolinae 10 (1969), 167-175. (Russian)
- [11] J. Dugundji: Topology, Allyn and Bacon, Inc., 1966.
- [12] J. Heinonen: Lectures on Lipschitz analysis, Report. University of Jyväskulä, Department of Mathematics and Statistics, 100. University of Jyväskulä, Jyväskulä, 2005.
- [13] H. Ishihara: On the constructive Hahn-Banach theorem, Bull. London. Math. Soc. 21, 1989, 79-81.
- [14] H. Ishihara: Constructive (functional) analysis, Lecture 3, Proof and Computation, Fischbachau, 3-8 October, 2019, slides.
- [15] W. Julian and K. Philips: Constructive Bounded Sequences and Lipschitz Functions, J. London Math. Soc., s2-31 (3), 1985, 385-392.
- [16] B. V. Limaye: Linear Functional Analysis for Scientists and Engineers, Springer Science+Business Media Singapore, 2016.

- [17] I. Loeb: Lipschitz functions in constructive reverse mathematics, Logic Journal of the IGPL (special issue on Non-Classical Mathematics), Volume 21, Number 1: 28-43, 2013.
- [18] M. Mandelkern: Constructive Continuity, Mem. Amer. Math. Soc. 277, 1983.
- [19] E. J. McShane: Extension of range of functions, Bull. Amer. Math. Soc. 40, 1934, no.12, 837-842.
- [20] J. Myhill: Constructive Set Theory, J. Symbolic Logic 40, 1975, 347-382.
- [21] I. Petrakis: A direct constructive proof of a Stone-Weierstrass theorem for metric spaces, in A. Beckmann, L. Bienvenu and N. Jonoska (Eds.) Pursuit of the Universal, CiE 2016, Springer, LNCS 9709, 364-374, 2016.
- [22] I. Petrakis: McShane-Whitney pairs, in J. Kari, F. Manea and I. Petre (Eds.): Unveiling Dynamics and Complexity, CiE 2017, LNCS 10307, 351-363.
- [23] S. G. Simpson: Subsystems of Second Order Arithmetic, Second Edition, Cambridge University Press, 2009.
- [24] H. Tuominen: Analysis in Metric Spaces, Lecture notes, 2014.
- [25] N. Weaver: Lipschitz Algebras, World Scientific, 1999.
- [26] H. Whitney: Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36, 1934, no. 1, 63-89.
- [27] http://math.stackexchange.com/questions/665587/